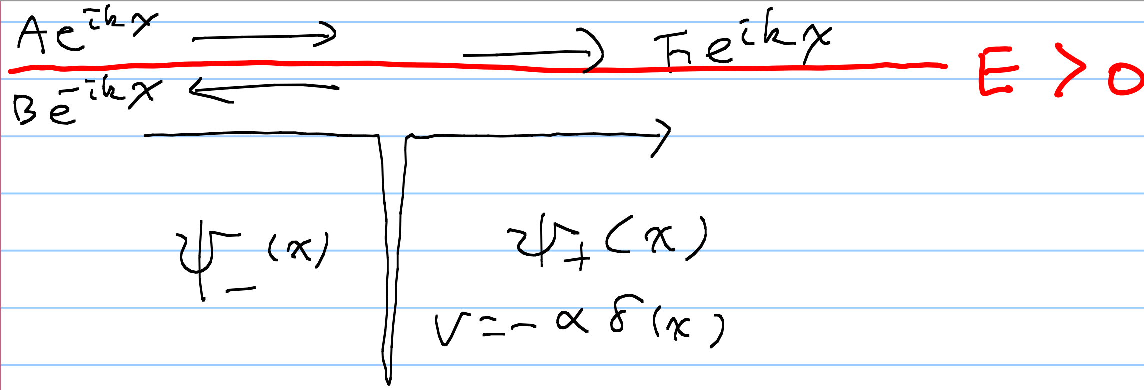
1D scattering and More

Note Title



From last lecture, we have found that

$$B = \frac{i\beta}{1-i\beta}A, \qquad F = \frac{A}{1-i\beta}$$

$$\text{with } \beta \equiv \frac{m\alpha}{\hbar^2 k}$$

So

$$\psi_-(x) = Ae^{ikx} + Be^{-ikx}$$
$$= A\left(e^{ikx} + \frac{i\beta}{1-i\beta}e^{-ikx}\right)$$
$$\psi_+(x) = Fe^{ikx} = \frac{A}{1-i\beta}e^{ikx}$$

Let's check how $|\psi_-(x)|^2$ & $|\psi_+(x)|^2$ behave.

In order to simplify the math, let's define

$$\frac{i\beta}{1-i\beta} \equiv r_1 e^{i\theta_1}, \qquad r_1 \equiv \left|\frac{i\beta}{1-i\beta}\right| = \sqrt{\frac{\beta^2}{1+\beta^2}}$$

$$\frac{1}{1-i\beta} \equiv r_2 e^{i\theta_2}, \qquad r_2 \equiv \left|\frac{1}{1-i\beta}\right| = \sqrt{\frac{1}{1+\beta^2}}$$

Then

$$\psi_-(x) = A\left(e^{ikx} + r_1 e^{-i(kx-\theta_1)}\right)$$
$$\psi_+(x) = A r_2 e^{i(kx+\theta_2)}$$

$$\Rightarrow |\psi_-(x)|^2 = |A|^2\left(1 + r_1^2 + r_1 e^{-i(2kx-\theta_1)} + r_1 e^{i(2kx-\theta_1)}\right)$$

$$\Rightarrow |\psi_-(x)|^2 = |A|^2 \left(1 + r_1^2 + 2 r_1 \cos(2kx - \theta_1) \right)$$

Since $\cos(2kx - \theta_1)$ varies from -1 to $+1$, as x changes

$$|\psi_-(x)|^2_{max} = |A|^2 (1 + r_1^2 + 2r_1) = |A|^2 (1 + r_1)^2$$

$$|\psi_-(x)|^2_{min} = |A|^2 (1 + r_1^2 - 2r_1) = |A|^2 (1 - r_1)^2$$

Similarly for $|\psi_+(x)|^2$

$$|\psi_+(x)|^2 = |A|^2 r_2^2$$, which is constant regardless of x

Noting that the actual **wave ftn** $\Psi(x,t)$ is given by

$$\Psi_-(x,t) = A e^{ikx} \cdot e^{-i\frac{E}{\hbar}t} + B e^{-ikx} e^{-i\frac{E}{\hbar}t}$$
$$= \psi_-(x) e^{-i\frac{E}{\hbar}t}$$

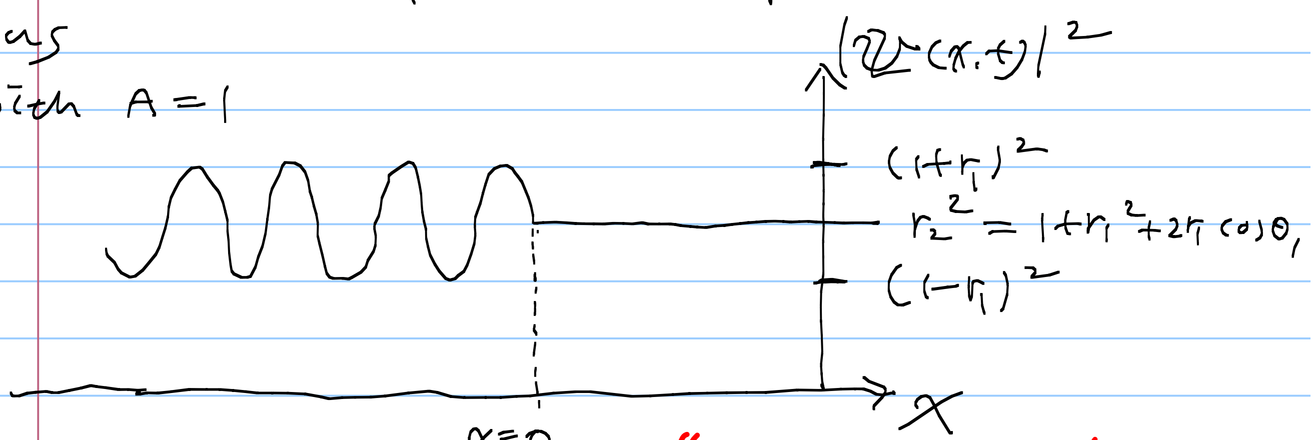
$$\Psi_+(x,t) = \psi_+(x) e^{-i\frac{E}{\hbar}t}$$

Note: $\Psi(x,t)$ is the actual **wave function** and $\psi(x)$ is just the **eigen function.**

We find that $|\Psi(x,t)|^2 = |\psi(x)|^2$ for this case.

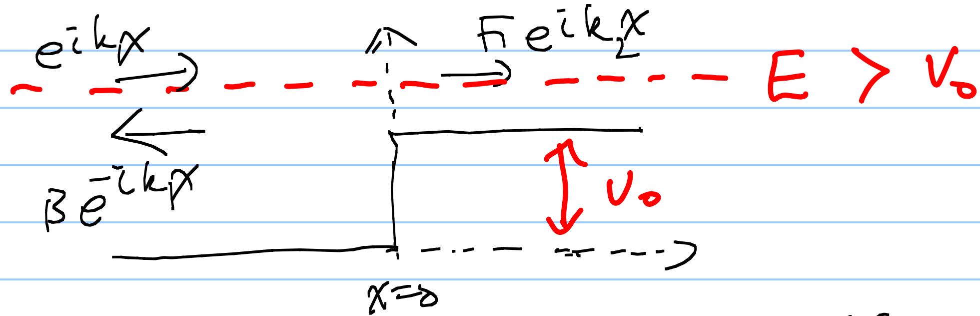
Thus

with $A=1$



"At any time"

Step potential ($V(x) = V_0\,\theta(x)$)



for $x<0$

$$-\frac{\hbar^2}{2m}\psi_-''(x) = E\,\psi_-(x) \Rightarrow k_1 = \frac{\sqrt{2mE}}{\hbar}$$

$$-\frac{\hbar^2}{2m}\psi_+''(x) + V_0\,\psi_+(x) = E\,\psi_+(x) \Rightarrow k_2 = \frac{\sqrt{2m(E-V_0)}}{\hbar}$$

for $x>0$

$$\Downarrow$$

$$k_1 > k_2$$

$$\psi_-(x) = e^{ik_1x} + Be^{-ik_1x}$$

$$\psi_+(x) = F\,e^{ik_2x}$$

Boundary conditions

① $\psi(x)$ continuous

② $\frac{d\psi(x)}{dx}$ continuous, ∵ no infinite potential

with ①, $1 + B = F$

with ②, $ik_1(1-B) = ik_2F$

$$\Rightarrow 1 - B = \frac{k_2}{k_1}F$$

$$\Rightarrow 2 = \left(1 + \frac{k_2}{k_1}\right)F \Rightarrow F = \frac{2}{1+\frac{k_2}{k_1}}$$

$$B = F - 1 = \frac{2}{1+\frac{k_2}{k_1}} - 1 = \frac{1 - k_2/k_1}{1 + k_2/k_1}$$

$$R = \frac{k_1 |B|^2}{k_1 |A|^2} = |B|^2 = \left| \frac{k_1 - k_2}{k_1 + k_2} \right|^2$$

$$T = \frac{k_2 |F|^2}{k_1 |A|^2} = \frac{k_2}{k_1} \cdot \left| \frac{2}{1 + \frac{k_2}{k_1}} \right|^2 = \frac{k_2}{k_1} \frac{4 k_1^2}{(k_1 + k_2)^2}$$

$$= \frac{4 k_1 k_2}{(k_1 + k_2)^2}$$

$$R + T = \frac{(k_1 + k_2)^2}{(k_1 + k_2)^2} = 1$$ as it should be.

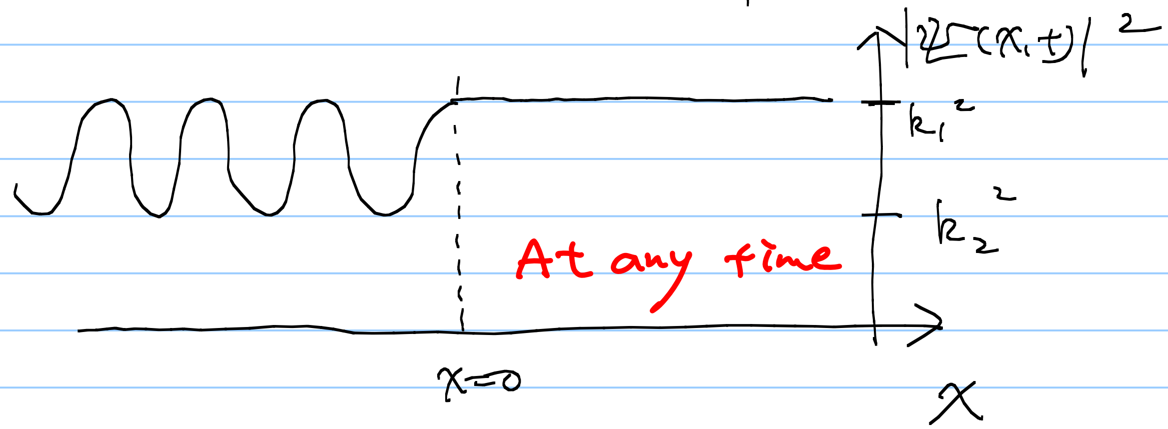
Now $$|\psi_-(x)|^2 = 1 + B^2 + B\left(e^{-2ik_1 x} + e^{+2ik_1 x}\right)$$

$$= 1 + B^2 + 2B\cos(k_1 x)$$

$$|\psi_-(x)|^2_{max} = (1+B)^2 = F^2 = \frac{4k_1^2}{(k_1 + k_2)^2}$$

$$|\psi_-(x)|^2_{min} = (1-B)^2 = \left(\frac{k_2}{k_1}\right)^2 F^2 = \frac{4k_2^2}{(k_1+k_2)^2}$$

$$|\psi_+(x)|^2 = |F|^2 = \left| \frac{2}{1 + \frac{k_2}{k_1}} \right|^2 = \frac{4k_1^2}{(k_1+k_2)^2}$$

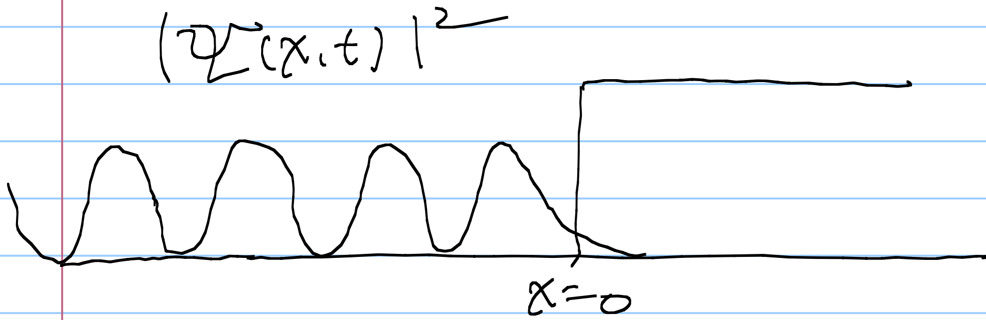


Interestingly, the probability of finding the particle on the positive side (on the step potential) is never smaller than the value on the negative x side.

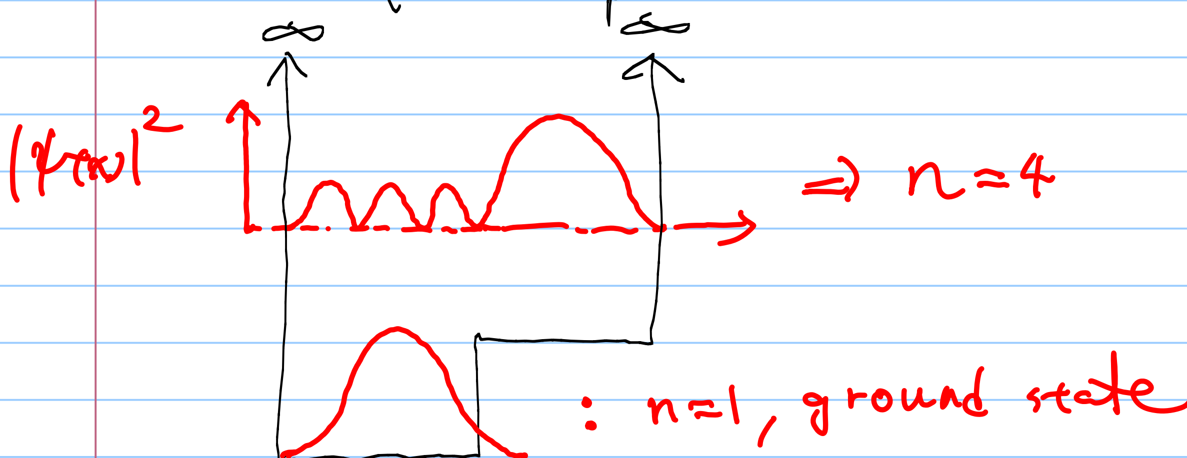
Now for $E < V_0$:

Do this as a homework

partial answer



A related example:



Remember: "Wave ftn should have time dependence"

- Wave function $\Psi(x,t)$ is the solution of the (time-dependent) Schrödinger Eq.
- $\psi(x)$ is NOT a wavefunction; it is just an eigenfunction of the time-independent Schrödinger Eq.
- $\psi(x)\, e^{-i\frac{E}{\hbar}t}$ IS a valid wave function.